

THE HAGEDORN STRUCTURE OF THE NON-PERTURBATIVE GLUON PRESSURE WITHIN THE MASS GAP APPROACH TO QCD

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(Dated: April 6, 2016)

We have shown in detail that the low-temperature expansion for the non-perturbative gluon pressure has the Hagedorn-type structure. Its exponential spectrum of all the effective gluonic excitations are expressed in terms of the mass gap. It is this which is responsible for the large-scale dynamical structure of the QCD ground state. The gluon pressure properly scaled has a maximum at some characteristic temperature $T = T_c = 266.5$ MeV, separating the low- and high temperature regions. The gluon pressure is exponentially suppressed in the $T \rightarrow 0$ limit. In the $T \rightarrow T_c$ limit it demonstrates an exponential rise in the number of dynamical degrees of freedom. This makes it possible to identify T_c with the Hagedorn transition temperature T_h , i.e., to put $T_h = T_c$. The gluon pressure has a complicated dependence on the mass gap and temperature near T_c and up to approximately $(4 - 5)T_c$. In the limit of very high temperatures $T \rightarrow \infty$ its polynomial character is confirmed, containing the terms proportional to T^2 and T .

PACS numbers: 11.10.Wx, 12.38.Mh, 12.38.Lg, 12.38.Aw

I. INTRODUCTION

The properties of Quantum Chromodynamics (QCD) at finite temperature and density are subject to the intense investigations by lattice and analytic methods [1] (and references therein). The effective potential approach for composite operators [2] turned out to be effective and perspective analytical tool for the generalization of QCD to non-zero temperature and density. In the absence of external sources it is nothing but the vacuum energy density (VED), i.e., the pressure apart from the sign. This approach is non-perturbative (NP) from the very beginning, since it deals with the expansion of the corresponding skeleton vacuum loop diagrams in powers of the Planck constant, and thus allows one to calculate the VED from first principles. In accordance with this program we have extended [3] to non-zero temperature in [4]. This made it possible to introduce the correctly defined temperature-dependent bag constant (bag pressure) as a function of the mass gap. It is this which is responsible for the large-scale dynamical structure of the QCD ground state [5] and coincides with the Jaffe-Witten (JW) mass gap [6] by properties. The confining dynamics in the gluon matter (GM) is therefore nontrivially taken into account directly through the mass gap and via the temperature-dependent bag constant itself, but other NP effects due to the mass gap are also present. Being NP, the effective approach for composite operators, nevertheless, makes it possible to incorporate the thermal perturbation theory (PT) expansion in a self-consistent way. In [5] we have formulated and developed the analytic thermal PT which allows one to calculate the PT contributions in terms of the convergent series in integer powers of a small α_s . We have also explicitly derived the first PT correction of the α_s -order to the NP part of the GM equation of state (EoS), which we call the NP gluon pressure, so it is understood as a main part of the full EoS.

In this article from the very beginning, we are investigating a system at non-zero temperature, which consists of $SU(3)$ purely Yang-Mills (YM) gauge fields without quark degrees of freedom (i.e., at zero density). Its primary aim is to explicitly show the Hagedorn structure [7, 8] of the above-mentioned gluon pressure below some characteristic temperature T_c . For its numerical value see Fig. 1 below. We argue that it has to be identified with the Hagedorn transition temperature T_h , i.e., $T_h = T_c$ within our approach. In our opinion, the Hagedorn character of the expansion for any pressure (calculated with the help of lattice or analytic methods) in the region of low temperatures is of crucial importance for correct understanding and description of the GM dynamical content. For this we begin with sections II, III, and IV in which we shortly describe our results obtained earlier in [4, 5]. They are present here for the readers convenience in order to have a general picture at hand and because the book [5], where we have summarized our previous results, is not freely available. In the main sections V and VI we have numerically calculated the gluon

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pressure and analytically derived its low-temperature expansion in detail, respectively. It is explicitly shown that it is nothing else but the Hagedorn-type exponential series for the effective gluonic excitations of various dynamical nature, which are expressed in terms of the mass gap only. Section VII is devoted to a brief discussion of the high-temperature expansion for the gluon pressure. In section VIII we discuss our conclusions.

II. THE GLUON PRESSURE AT NON-ZERO TEMPERATURE

In the imaginary-time formalism [9–11], all of the four-dimensional integrals can be easily generalized to non-zero temperature T according to the prescription (let us remind that in all our publications as well as in this paper the signature is Euclidean in order to avoid non-physical singularities at light-cone from the very beginning)

$$\int \frac{dq_0}{(2\pi)} \rightarrow T \sum_{n=-\infty}^{+\infty}, \quad q^2 = \mathbf{q}^2 + q_0^2 = \mathbf{q}^2 + \omega_n^2 = \omega^2 + \omega_n^2, \quad \omega_n = 2n\pi T. \quad (2.1)$$

In other words, each integral over q_0 of the loop momentum is to be replaced by the sum over the Matsubara frequencies labeled by n , which obviously assumes the replacement $q_0 \rightarrow \omega_n = 2n\pi T$ for bosons (gluons).

Introducing the temperature dependence into the gluon pressure [3–5], we obtain

$$P_g(T) = P_{NP}(T) + P_{PT}(T) = B_{YM}(T) + P_{YM}(T) + P_{PT}(T), \quad (2.2)$$

where the corresponding terms in frequency-momentum space are:

$$B_{YM}(T) = \frac{8}{\pi^2} \int_0^{\omega_{eff}} d\omega \, \omega^2 T \sum_{n=-\infty}^{+\infty} \left[\ln(1 + 3\alpha^{INP}(\omega^2, \omega_n^2)) - \frac{3}{4}\alpha^{INP}(\omega^2, \omega_n^2) \right], \quad (2.3)$$

$$P_{YM}(T) = -\frac{8}{\pi^2} \int_0^\infty d\omega \, \omega^2 T \sum_{n=-\infty}^{+\infty} \left[\ln\left(1 + \frac{3}{4}\alpha^{INP}(\omega^2, \omega_n^2)\right) - \frac{3}{4}\alpha^{INP}(\omega^2, \omega_n^2) \right], \quad (2.4)$$

$$P_{PT}(T) = -\frac{8}{\pi^2} \int_{\Lambda_{YM}}^\infty d\omega \, \omega^2 T \sum_{n=-\infty}^{+\infty} \left[\ln\left(1 + \frac{3\alpha^{PT}(\omega^2, \omega_n^2)}{4 + 3\alpha^{INP}(\omega^2, \omega_n^2)}\right) - \frac{3}{4}\alpha^{PT}(\omega^2, \omega_n^2) \right]. \quad (2.5)$$

In frequency-momentum space the intrinsically non-perturbative (INP) and PT effective charges become

$$\alpha^{INP}(q^2) = \frac{\Delta^2}{q^2} = \alpha^{INP}(\omega^2, \omega_n^2) = \frac{\Delta^2}{\omega^2 + \omega_n^2}, \quad (2.6)$$

and

$$\alpha^{PT}(q^2) = \frac{\alpha_s}{1 + \alpha_s b_0 \ln(q^2/\Lambda_{YM}^2)} = \alpha^{PT}(\omega^2, \omega_n^2) = \frac{\alpha_s}{1 + \alpha_s b_0 \ln(\omega^2 + \omega_n^2/\Lambda_{YM}^2)}, \quad (2.7)$$

respectively. It is also convenient to introduce the following notations:

$$T^{-1} = \beta, \quad \omega = \sqrt{\mathbf{q}^2}, \quad (2.8)$$

where, evidently, in all the expressions \mathbf{q}^2 is the square of the three-dimensional loop momentum, in complete agreement with the relations (2.1), and ω_{eff} is a scale separating the low- and high frequency-momentum regions.

In Eq. (2.6) Δ^2 is the mass gap, mentioned above, which is responsible for the large-scale dynamical structure of the QCD vacuum, and thus determines the scale of its NP dynamics. We have shown that confining effective charge (2.6), and hence its β -function, is a result of the summation of the skeleton (i.e., NP) loop diagrams, contributing to

the full gluon self-energy in the $q^2 \rightarrow 0$ limit (the strong coupling regime for the effective charge). This summation has been performed within the corresponding equation of motion. It has been done without violating the $SU(3)$ color gauge invariance of QCD [5] (and references therein). In more detail (including the interpretation of Eq. (2.6) and the explanation of all the notations above) the derivation of the bag constant as a function of the mass gap and its generalization to non-zero temperature has been completed in [3] and [4], respectively.

The PT effective charge $\alpha^{PT}(q^2)$ (2.7) is the generalization to non-zero temperature of the renormalization group equation solution, the so-called sum of the main PT logarithms [5, 12–14] (its analog as a function of the variable T/T_c (see below) can be found, for example in [5, 10, 15]). Here $\Lambda_{YM}^2 = 0.09 \text{ GeV}^2$ [16] is the asymptotic scale parameter for $SU(3)$ YM fields, and $b_0 = (11/4\pi)$ for these fields, while the strong fine-structure constant is $\alpha_s \equiv \alpha_s(m_Z) = 0.1184$ [17]. In Eq. (2.7) q^2 cannot go below Λ_{YM}^2 , i.e., $\Lambda_{YM}^2 \leq q^2 \leq \infty$, which has already been symbolically shown in Eq. (2.5). It is worth reminding that the separation between effective charges (2.6) and (2.7), is not only exact but it is unique one as well. It has been done by the subtraction of the PT part from the full gluon propagator with the respect of the mass gap, see [4, 5].

The NP pressure $P_{NP}(T) = B_{YM}(T) + P_{YM}(T)$ and the PT pressure $P_{PT}(T)$, and hence the gluon pressure $P_g(T)$ (2.2), are normalized to zero when the interaction is formally switched off, i.e., letting $\alpha_s = \Delta^2 = 0$. This means that the initial normalization condition of the free PT vacuum to zero holds at non-zero temperature as well.

III. $P_{NP}(T)$ CONTRIBUTION

One of the attractive features of the confining effective charge (2.6) is that it allows an exact summation over the Matsubara frequencies in the NP pressure $P_{NP}(T)$ given by the sum of the integrals (2.3) and (2.4). Collecting all the analytical results obtained in [4, 5], we can write

$$P_{NP}(T) = \frac{6}{\pi^2} \Delta^2 P_1(T) + \frac{16}{\pi^2} T M(T). \quad (3.1)$$

Here $P_1(T)$ and $M(T)$ are

$$P_1(T) = \int_{\omega_{eff}}^{\infty} d\omega \frac{\omega}{e^{\beta\omega} - 1}, \quad (3.2)$$

and

$$M(T) = [P_2(T) + P_3(T) - P_4(T)], \quad (3.3)$$

respectively, while

$$\begin{aligned} P_2(T) &= \int_{\omega_{eff}}^{\infty} d\omega \, \omega^2 \ln(1 - e^{-\beta\omega}), \\ P_3(T) &= \int_0^{\omega_{eff}} d\omega \, \omega^2 \ln(1 - e^{-\beta\omega'}), \\ P_4(T) &= \int_0^{\infty} d\omega \, \omega^2 \ln(1 - e^{-\beta\bar{\omega}}), \end{aligned} \quad (3.4)$$

and ω' and $\bar{\omega}$ are given by the relations

$$\omega' = \sqrt{\omega^2 + 3\Delta^2} = \sqrt{\omega^2 + m_{eff}^{\prime 2}}, \quad m_{eff}' = \sqrt{3}\Delta, \quad (3.5)$$

and

$$\bar{\omega} = \sqrt{\omega^2 + \frac{3}{4}\Delta^2} = \sqrt{\omega^2 + \bar{m}_{eff}^2}, \quad \bar{m}_{eff} = \frac{\sqrt{3}}{2}\Delta, \quad (3.6)$$

respectively. It is worth reminding that in the NP pressure (3.1) the bag pressure $B_{YM}(T)$ (2.3) is responsible for the formation of the massive gluonic excitations ω' (3.5), while the YM part $P_{YM}(T)$ (2.4) is responsible for the formation of the massive gluonic excitations $\bar{\omega}$ (3.6).

The so-called gluon mean number [9], also known as Bose-Einstein distribution, is

$$N_g \equiv N_g(\beta, \omega) = \frac{1}{e^{\beta\omega} - 1}, \quad (3.7)$$

where β and ω are defined in Eq. (2.8). It appears in the integrals (3.3)-(3.4) and describes the distribution and correlation of massless gluons in the medium. Replacing ω by $\bar{\omega}$ and ω' we can consider the corresponding gluon mean numbers as describing the distribution and correlation of the corresponding massive gluonic excitations in the medium, see integrals $P_3(T)$ and $P_4(T)$ in Eqs. (3.4). They are of NP dynamical origin, since their masses are due to the mass gap Δ^2 . All three different gluon mean numbers range continuously from zero to infinity [9]. We have the two different massless excitations, propagating in accordance with the integral (3.2) and the first of the integrals (3.4). However, they are not free, since in the PT $\Delta^2 = 0$ limit they vanish (the composition (3.3) becomes zero in this case). So the NP pressure describes the four different effective gluonic excitations. The gluon mean numbers are closely related to the thermodynamic observables, especially to the pressure. Its exponential suppression in the $T \rightarrow 0$ limit and the polynomial structure in the $T \rightarrow \infty$ limit are determined by the corresponding asymptotics of the gluon mean numbers, see below.

Concluding, let us emphasize that the effective scale ω_{eff} is not an independent scale parameter. Due to extremization of the mass gap-dependent effective potential, from the stationary condition at zero temperature [3] and the scale-setting scheme at non-zero temperature [4] it follows that

$$\omega_{eff} = 1.48\Delta, \quad \Delta = 0.6756 \text{ GeV}. \quad (3.8)$$

So it is expressed in terms of the initial fundamental and unique mass scale parameter in our approach - the mass gap Δ (for simplicity, its squared version Δ^2 is conventionally called the mass gap as well throughout this paper). The introduction of ω_{eff} is also convenient from the technical point of view in order to simplify our expressions, which otherwise would be rather cumbersome (see below).

IV. THERMAL PT

One of our primary goals in [5] was to develop the analytic formalism for the numerical calculation of the PT term (2.5). It made it possible to calculate the PT contribution (2.5) to the gluon pressure (2.2) in terms of the convergent series in integer powers of a small α_s . For this goal, it is convenient to re-write the integral (2.5) as follows:

$$P_{PT}(T) = -\frac{8}{\pi^2} \int_{\Lambda_{YM}}^{\infty} d\omega \, \omega^2 T \sum_{n=-\infty}^{+\infty} \left[\ln[1 + x(\omega^2, \omega_n^2)] - \frac{3}{4} \alpha^{PT}(\omega^2, \omega_n^2) \right], \quad (4.1)$$

where

$$x(\omega^2, \omega_n^2) = \frac{3\alpha^{PT}(\omega^2, \omega_n^2)}{4 + 3\alpha^{INP}(\omega^2, \omega_n^2)} = \frac{3}{4} \frac{(\omega^2 + \omega_n^2)}{M(\bar{\omega}^2, \omega_n^2)} \frac{\alpha_s}{(1 + \alpha_s \ln z_n)} \quad (4.2)$$

with the help of the expressions (2.6) and (2.7), and where

$$M(\bar{\omega}^2, \omega_n^2) = \bar{\omega}^2 + \omega_n^2, \quad \ln z_n \equiv \ln z(\omega^2, \omega_n^2) = b_0 \ln[(\omega^2 + \omega_n^2)/\Lambda_{YM}^2], \quad (4.3)$$

and $\bar{\omega}^2$ is given in Eq. (3.5). Let us also note that in these notations

$$\alpha^{PT}(\omega^2, \omega_n^2) \equiv \alpha(z_n) = \frac{\alpha_s}{(1 + \alpha_s \ln z_n)}. \quad (4.4)$$

Collecting all the results obtained in [5], where it has been explicitly shown that variable $x(\omega^2, \omega_n^2)$ always is very small, we are able to present the PT part of the gluon pressure as a sum of the two terms, namely

$$P_{PT}(T) = P_{PT}(\Delta^2; T) + O(\alpha_s^2), \quad (4.5)$$

where

$$P_{PT}(\Delta^2; T) = \sum_{k=1}^{\infty} \alpha_s^k P_k(\Delta^2; T) \quad (4.6)$$

with

$$P_k(\Delta^2; T) = \frac{9}{2\pi^2} \Delta^2 \int_{\Lambda_{YM}} d\omega \, \omega^2 T \sum_{n=-\infty}^{+\infty} \left[\frac{1}{M(\bar{\omega}^2, \omega_n^2)} (-1)^{k-1} \ln^{k-1} z_n \right]. \quad (4.7)$$

Here $P_{PT}(\Delta^2; T)$ (4.6) describes the Δ^2 -dependent PT contribution to the NP term $P_{NP}(T)$ (3.1), beginning with the α_s -order term. In fact, the whole expansion (4.6) is the correction in integer powers of α_s to the NP term $P_{NP}(T)$ (3.1), i.e., to call it the PT term is only convention. The α_s^2 -order term is also a sum of the two terms, one of which depends on the mass gap and the other one does not. The corresponding convergent expansions for them in integer powers of a small $x(\omega^2, \omega_n^2)$ (or, equivalently, α_s) begin with α_s^2 -order terms, see [5]. They are not shown explicitly, since numerically they are much smaller than the first term in Eq. (4.5). For this reason their consideration will be omitted in what follows. In other words, we put in Eq. (4.5) $P_{PT}(T) = P_{PT}(\Delta^2; T) = P_{PT}^s(T) = \alpha_s P_1(\Delta^2; T)$, on account of the series (4.6) up to α_s^2 -order, as underlined above. Concluding, let us note that the convergence of series (4.6) has been proven in [5] as well.

V. THE GLUON PRESSURE $P_g(T)$

Taking into account the above-mentioned remarks, the gluon pressure (2.2) then becomes

$$P_g(T) = P_{NP}(T) + P_{PT}^s(T) = \frac{6}{\pi^2} \Delta^2 P_1(T) + \frac{16}{\pi^2} T M(T) + P_{PT}^s(T), \quad (5.1)$$

on account of Eqs. (3.1)-(3.4). In the integral (4.7) for $k = 1$ the summation over the Matsubara frequencies can be performed analytically, i.e., exactly [4, 5]. So finally for $P_{PT}^s(T)$, one obtains

$$P_{PT}^s(T) \equiv P_{PT}^s(\Delta^2; T) = \alpha_s \times \frac{9}{2\pi^2} \Delta^2 \int_{\Lambda_{YM}} d\omega \, \omega^2 \frac{1}{\bar{\omega}} \frac{1}{e^{\beta\bar{\omega}} - 1}. \quad (5.2)$$

Here it is worth noting only that the PT term (5.2) describes the same massive gluonic excitations $\bar{\omega}$ (3.5), but their propagation, however, suppressed by the α_s -order. We can consider it as a new massive excitation in the GM, denoted it as $\alpha_s \cdot \bar{\omega}$. Let us remind once more that the term $P_{PT}^s(T)$ is NP, depending on the mass gap Δ^2 , which is only suppressed by the α_s order. In the PT $\Delta^2 = 0$ limit, the above-defined composition $M(T)$ becomes zero, as it follows from Eqs. (3.3)-(3.4), and thus the gluon pressure $P_g(T)$ itself shown in (5.1). So it is truly NP quantity, indeed, in agreement with the normalization condition of the free PT vacuum to zero.

Numerically calculated the gluon pressure (5.1) is shown in Fig. 1. It has a maximum at some "characteristic" temperature $T = T_c = 266.5$ MeV. It is necessary to investigate the low-temperature (below T_c) behavior of the gluon pressure (5.1) in detail. It will make it possible to explicitly show the Hagedorn nature of the corresponding expansion in this region. At the same time, its high-temperature (above T_c) behavior suffices to briefly discuss.

VI. LOW-TEMPERATURE EXPANSION. THE HAGEDORN STRUCTURE

In order to investigate the behavior of the gluon pressure (5.1) in the low-temperature region ($T \leq T_c$) let us note that in the integrals (3.2), (3.4) and (5.2) the variable $e^{-\beta\omega}$ with the replacements $\omega \rightarrow \omega', \bar{\omega}$ is always small in this region, especially in the ($T \rightarrow 0, \beta = T^{-1} \rightarrow \infty$) limit. So one can expand the corresponding mean numbers (3.6) in the form of the corresponding Taylor series [18] as follows:

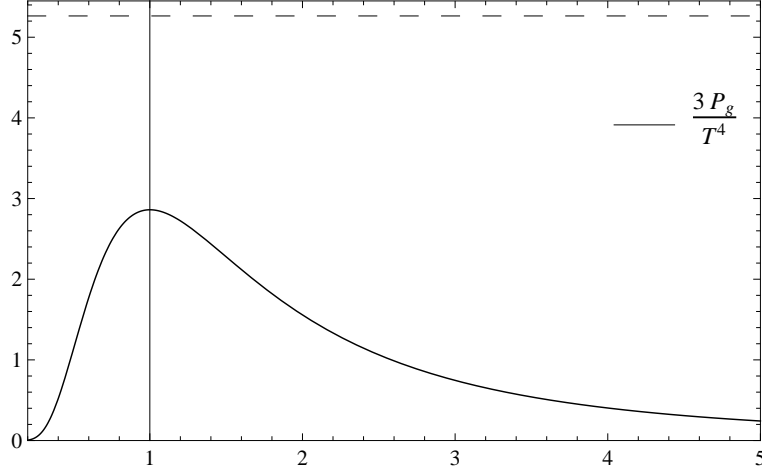


FIG. 1: The gluon pressure (5.1) scaled (i.e., divided) by $T^4/3$ is shown as a function of T/T_c (solid curve). It has a maximum at $T = T_c = 266.5$ MeV (vertical solid line). The horizontal dashed line is the general Stefan-Boltzmann (SB) constant $3P_{SB}(T)/T^4 = (24/45)\pi^2$. One can conclude that NP effects due to the mass gap are still important approximately up to $5T_c$.

$$N_g \equiv N_g(\beta, \omega) = \frac{1}{e^{\beta\omega} - 1} = e^{-\beta\omega}(1 - e^{-\beta\omega})^{-1} = \sum_{n=1}^{\infty} e^{-n\beta\omega} \quad (6.1)$$

and

$$\ln(1 - e^{-\beta\omega}) = - \sum_{n=1}^{\infty} \frac{1}{n} e^{-n\beta\omega} \quad (6.2)$$

with the above-mentioned replacements (here n is different from n in Eqs. (2.3)-(2.5)). After substitution of these series into the corresponding integrals, such obtained terms can be explicitly integrated termwise, since the Taylor series (6.1) and (6.2) are convergent in this temperature region and integrals calculated in this section are not divergent.

Let us begin with pointing out in advance that all exactly calculated integrals, discussed below can be found in [18, 19]. So the integral $P_1(T)$ defined in Eq. (3.2) becomes

$$P_1(T) = \int_{\omega_{eff}}^{\infty} d\omega \, \omega \, N_g(\beta, \omega) = \int_{\omega_{eff}}^{\infty} d\omega \, \omega \sum_{n=1}^{\infty} e^{-n\beta\omega} = \sum_{n=1}^{\infty} \int_{\omega_{eff}}^{\infty} d\omega \, \omega e^{-n\beta\omega}. \quad (6.3)$$

The almost trivial integration yields

$$P_1(T) = \sum_{n=1}^{\infty} \left(\frac{1}{n^2} T^2 + \frac{1}{n} \omega_{eff} T \right) e^{-n \frac{\omega_{eff}}{T}}. \quad (6.4)$$

The integral $P_2(T)$ defined in Eqs. (3.4) can be considered in the same way after the substitution of the expansion (6.2), so it becomes

$$P_2(T) = \int_{\omega_{eff}}^{\infty} d\omega \, \omega^2 \ln(1 - e^{-\beta\omega}) = - \sum_{n=1}^{\infty} \frac{1}{n} \int_{\omega_{eff}}^{\infty} d\omega \, \omega^2 e^{-n\beta\omega}, \quad (6.5)$$

and exactly integrating it, one obtains

$$P_2(T) = - \sum_{n=1}^{\infty} \frac{1}{n} \left(\frac{2}{n^3} T^3 + \frac{2}{n^2} \omega_{eff} T^2 + \frac{1}{n} \omega_{eff}^2 T \right) e^{-n \frac{\omega_{eff}}{T}}. \quad (6.6)$$

The integral $P_3(T)$ defined in Eqs. (3.4) after the substitution of the expansion (6.2) with the replacement $\omega \rightarrow \omega'$ looks like

$$P_3(T) = \int_0^{\omega_{eff}} d\omega \, \omega^2 \ln(1 - e^{-\beta\omega'}) = - \sum_{n=1}^{\infty} \frac{1}{n} \int_0^{\omega_{eff}} d\omega \, \omega^2 e^{-n\beta\omega'}. \quad (6.7)$$

Replacing the variable ω by the variable ω' in accordance with the relation (3.5), this integral becomes

$$P_3(T) = - \sum_{n=1}^{\infty} \frac{1}{n} \int_a^{\omega'_{eff}} d\omega' \, \omega' \sqrt{(\omega'^2 - a^2)} e^{-n\beta\omega'}, \quad (6.8)$$

where

$$\omega'_{eff} = \sqrt{(\omega_{eff}^2 + a^2)}, \quad a = \sqrt{3}\Delta. \quad (6.9)$$

Noting further that the variable $x = a^2/\omega'^2 \leq 1$, we can formally expand

$$\sqrt{(\omega'^2 - a^2)} = \omega' (1 - x)^{1/2} = \omega' \left[1 - \frac{1}{2} \frac{a^2}{\omega'^2} + \sum_{k=2}^{\infty} \binom{1/2}{k} (-x)^k \right], \quad (6.10)$$

then from the last integral it follows

$$P_3(T) = - \sum_{n=1}^{\infty} \frac{1}{n} \int_a^{\omega'_{eff}} d\omega' \, \omega'^2 e^{-n\beta\omega'} + \frac{3}{2} \Delta^2 \sum_{n=1}^{\infty} \frac{1}{n} \int_a^{\omega'_{eff}} d\omega' \, e^{-n\beta\omega'} - \sum_{n=1}^{\infty} \frac{1}{n} P_3^{(n)}(T), \quad (6.11)$$

where

$$P_3^{(n)}(T) = - \int_a^{\omega'_{eff}} d\omega' \, \omega'^2 e^{-n\beta\omega'} \sum_{k=2}^{\infty} \binom{1/2}{k} (-x)^k. \quad (6.12)$$

Let us consider the last integral (6.12) in more detail. Since the series over k are convergent in the interval of integration and the functions depending on k are integrable in this interval, these series may be integrated termwise [18], that is,

$$P_3^{(n)}(T) = - \sum_{k=2}^{\infty} \binom{1/2}{k} (-a^2)^k \int_a^{\omega'_{eff}} d\omega' \, \frac{e^{-n\beta\omega'}}{(\omega')^{2k-2}}. \quad (6.13)$$

Integrating it, one obtains

$$P_3^{(n)}(T) = - \sum_{k=2}^{\infty} \binom{1/2}{k} (-a^2)^k \left[N_3^{(n,k)}(T, \omega') \right]_a^{\omega'_{eff}} \quad (6.14)$$

and $\left[N_3^{(n,k)}(T, \omega') \right]_a^{\omega'_{eff}}$ denotes the result of the integration over ω' in Eq. (6.13) in the interval $[a, \omega'_{eff}]$, while the function $N_3^{(n,k)}(T, \omega')$ itself is

$$N_3^{(n,k)}(T, \omega') = -e^{-n\beta\omega'} \sum_{m=1}^{2k-3} \frac{(-n\beta)^{m-1} (\omega')^{m+2-2k}}{(2k-3)(2k-4)\dots(2k-2-m)} + \frac{(-n\beta)^{2k-3}}{(2k-3)!} \text{Ei}(-n\beta\omega'), \quad k = 2, 3, 4, \dots \quad (6.15)$$

The series for the exponential integral function $\text{Ei}(-n\beta\omega')$ is [18]

$$\text{Ei}(-n\beta\omega') = e^{-n\beta\omega'} \sum_{l=1}^p (-1)^l \frac{(l-1)!}{(n\beta\omega')^l} + R_p, \quad (6.16)$$

where the relative error in the expansion (6.16) should satisfy $|R_p| < p!/(n\beta\omega')^{p+1}$ for real numbers. If one chooses $p = 2k - 4$ in the previous equation and correspondingly adjusting the relative error R_p , it is easy to show that both terms in Eq. (6.15) for $N_3^{(n,k)}(T, \omega')$ cancel each other termwise for any $k \geq 2$, and thus

$$N_3^{(n,k)}(T, \omega') = 0, \quad k = 2, 3, 4, \dots \quad (6.17)$$

which leads to

$$P_3^{(n)}(T) = 0 \quad (6.18)$$

via Eq. (6.14). Equivalently, we can choose $p = 2k - 3$ and neglecting R_p , then both terms in Eq. (6.15) for $N_3^{(n,k)}(T, \omega')$ again will cancel each other termwise for any $k \geq 2$, by neglecting the term of the same order as R_p in the first sum of Eq. (6.15). Going back to Eq. (6.11) and easily integrating the first two terms, and taking into account the previous result, one comes to the following expression, namely

$$\begin{aligned} P_3(T) &= \sum_{n=1}^{\infty} \frac{1}{n} \left(\frac{2}{n^3} T^3 + \frac{2}{n^2} \omega'_{eff} T^2 + \frac{1}{n} \omega'^2_{eff} T \right) e^{-n \frac{\omega'_{eff}}{T}} - \sum_{n=1}^{\infty} \frac{1}{n} \left(\frac{2}{n^3} T^3 + \frac{2}{n^2} a T^2 + \frac{1}{n} a^2 T \right) e^{-n \frac{a}{T}} \\ &\quad - \frac{1}{2} a^2 T \sum_{n=1}^{\infty} \frac{1}{n^2} \left(e^{-n \frac{\omega'_{eff}}{T}} - e^{-n \frac{a}{T}} \right). \end{aligned} \quad (6.19)$$

The integral $P_4(T)$ defined in Eqs. (3.4) after the substitution of the expansion (6.2) with the replacement $\omega \rightarrow \bar{\omega}$ looks like

$$P_4(T) = \int_0^{\infty} d\omega \, \omega^2 \ln(1 - e^{-\beta\omega}) = - \sum_{n=1}^{\infty} \frac{1}{n} \int_0^{\infty} d\omega \, \omega^2 e^{-n\beta\omega}, \quad (6.20)$$

and replacing the variable ω by the variable $\bar{\omega}$ in accordance with the relation (3.6), this integral becomes

$$P_4(T) = - \sum_{n=1}^{\infty} \frac{1}{n} \int_{(a/2)}^{\infty} d\bar{\omega} \, \bar{\omega} \sqrt{(\bar{\omega}^2 - (a/2)^2)} e^{-n\beta\bar{\omega}}. \quad (6.21)$$

Comparing Eq. (6.8) with this Eq. (6.21), one can conclude that the last one is the first one by putting formally $\omega'_{eff} = \infty$ and replacing $a \rightarrow a/2$. Doing so in the expansion (6.19), for integral (6.21) one finally obtains

$$P_4(T) = - \sum_{n=1}^{\infty} \frac{1}{n} \left(\frac{2}{n^3} T^3 + \frac{a}{n^2} T^2 + \frac{a^2}{4n} T \right) e^{-n \frac{a}{2T}} + \frac{1}{8} a^2 T \sum_{n=1}^{\infty} \frac{1}{n^2} e^{-n \frac{a}{2T}}. \quad (6.22)$$

Let us now consider Eq. (5.2), which after the substitution of the expansion (6.1) with the replacement $\omega \rightarrow \bar{\omega}$ becomes

$$P_{PT}^s(T) = \frac{9\alpha_s}{2\pi^2} \Delta^2 \int_{\Lambda_{YM}} d\omega \, \omega^2 \frac{1}{\bar{\omega} e^{\beta\bar{\omega}} - 1} = \frac{9\alpha_s}{2\pi^2} \Delta^2 \sum_{n=1}^{\infty} \int_{\Lambda_{YM}} d\omega \, \omega^2 \frac{1}{\bar{\omega}} e^{-n\beta\bar{\omega}}, \quad (6.23)$$

and $\bar{\omega}$ is given by the relation (3.6). Replacing the variable ω by the variable $\bar{\omega}$, one obtains

$$P_{PT}^s(T) = \frac{9\alpha_s}{2\pi^2} \Delta^2 \sum_{n=1}^{\infty} \int_{\tilde{\omega}_{eff}}^{\infty} d\tilde{\omega} \sqrt{(\tilde{\omega}^2 - (a/2)^2)} e^{-n\beta\tilde{\omega}}, \quad \tilde{\omega}_{eff} = \sqrt{\Lambda_{YM}^2 + (a/2)^2}, \quad (6.24)$$

and for a see Eq. (6.9). Noting that the variable $z = a^2/4\tilde{\omega}^2 < 1$ in this case, we can use the expansion like (6.10), taking into account only the substitution $a \rightarrow a/2$, in order to obtain

$$P_{PT}^s(T) = \frac{9\alpha_s}{2\pi^2} \Delta^2 \sum_{n=1}^{\infty} \left[\int_{\tilde{\omega}_{eff}}^{\infty} d\tilde{\omega} \tilde{\omega} e^{-n\beta\tilde{\omega}} - \frac{1}{8} a^2 \int_{\tilde{\omega}_{eff}}^{\infty} d\tilde{\omega} \frac{e^{-n\beta\tilde{\omega}}}{\tilde{\omega}} + P_s^{(n)}(T) \right]. \quad (6.25)$$

Due to the same formalism which has been used previously in order to get the result (6.18), one can conclude that $P_s^{(n)}(T) = 0$ as well. Easily integrating the first two terms, one comes to the following expansion

$$P_{PT}^s(T) = \frac{9\alpha_s}{2\pi^2} \Delta^2 \sum_{n=1}^{\infty} \left[\left(\frac{1}{n^2} T^2 + \frac{1}{n} \tilde{\omega}_{eff} T \right) e^{-n \frac{\tilde{\omega}_{eff}}{T}} + \frac{1}{8} a^2 \text{Ei} \left(-n \frac{\tilde{\omega}_{eff}}{T} \right) \right], \quad (6.26)$$

where the corresponding exponential integral function is defined by Eq. (6.16).

Collecting all our results of the corresponding integrations and after some re-arrangement of the terms, as well as introducing some new definitions (see below), one finally obtains

$$\begin{aligned} P_g(T) = & \frac{6}{\pi^2} \Delta^2 T^2 \sum_{n=1}^{\infty} \frac{1}{n^2} \left[\left(1 + \frac{n\omega_{eff}}{T} \right) e^{-n(\omega_{eff}/T)} - 4 \left(e^{-n(\omega'_{eff}/T)} - e^{-n(m'_{eff}/T)} \right) - e^{-n(\bar{m}_{eff}/T)} \right] \\ & + \frac{16}{\pi^2} T^4 \sum_{n=1}^{\infty} \frac{1}{n^2} \left[\left(\frac{2}{n^2} + \frac{2\omega'_{eff}}{nT} + \frac{\omega'^2_{eff}}{T^2} \right) e^{-n(\omega'_{eff}/T)} - \left(\frac{2}{n^2} + \frac{2\omega_{eff}}{nT} + \frac{\omega^2_{eff}}{T^2} \right) e^{-n(\omega_{eff}/T)} \right] \\ & - \frac{16}{\pi^2} T^4 \sum_{n=1}^{\infty} \frac{1}{n^2} \left[\left(\frac{2}{n^2} + \frac{2m'_{eff}}{nT} + \frac{m'^2_{eff}}{T^2} \right) e^{-n(m'_{eff}/T)} + \left(\frac{2}{n^2} + \frac{2\bar{m}_{eff}}{nT} + \frac{\bar{m}^2_{eff}}{T^2} \right) e^{-n(\bar{m}_{eff}/T)} \right] \\ & + \frac{9}{2\pi^2} \alpha_s \Delta^2 T^2 \sum_{n=1}^{\infty} \frac{1}{n^2} \left[\left(1 + \frac{n\tilde{\omega}_{eff}}{T} \right) e^{-n(\tilde{\omega}_{eff}/T)} + \frac{n^2 \bar{m}^2_{eff}}{2T^2} \text{Ei} \left(-n(\tilde{\omega}_{eff}/T) \right) \right], \quad T \leq T_c, \end{aligned} \quad (6.27)$$

where ω_{eff} is defined in (3.8). The effective masses m'_{eff} and \bar{m}_{eff} are defined in (3.5) and (3.6), respectively, while

$$\omega'_{eff} = \sqrt{\omega_{eff}^2 + 3\Delta^2}, \quad \tilde{\omega}_{eff} = \sqrt{\Lambda_{YM}^2 + (3/4)\Delta^2}. \quad (6.28)$$

The expression (6.27) is nothing else but the Hagedorn-type expansion [7, 8] for the gluon pressure in terms of the effective gluonic excitations of different dynamical nature in the low-temperature region. For example, reminding that the numerical value of the mass gap is $\Delta = 0.6756$ GeV, Eq. (3.8), the effective mass $\bar{m}_{eff} = 0.585$ GeV. It is comparable with an effective gluon mass of about (500 – 800) MeV [20], which one can conventionally interpret as the YM Debye screening mass. The effective mass $m'_{eff} = 1.17$ GeV, and it is comparable with masses of low-lying, lightest glueballs [20–23]. Thus, the corresponding gluonic excitations ω' (3.5) and $\tilde{\omega}$ (3.6) can be interpreted as the massive effective gluonic excitations. It is important to understand that the above-mentioned effective masses are not the pole masses which may appear in the gauge field propagators. This means that we cannot assign to the corresponding massive excitations a meaning of being physical particles. They have to be treated rather as quasi-particles, since they appear through the corresponding gluon mean numbers, something like the quark chemical potentials. Indeed, from Eq. (3.7) one gets

$$N'_g \equiv N_g(\beta, \omega') = \frac{1}{e^{\beta\omega'} - 1} = \frac{1}{e^{\beta\sqrt{\omega^2 + m'^2_{eff}}} - 1} = \frac{1}{e^{\beta(\omega - \mu'_g)} - 1}, \quad (6.29)$$

where we introduced the fictitious gluon "chemical potential" μ'_g . It has to satisfy the following equation $\mu'^2_g - 2\omega\mu'_g - m'^2_{eff} = 0$, which has the two independent solutions: $\mu'_g = \omega \pm \omega' = \omega \pm \sqrt{\omega^2 + m'^2_{eff}}$, leading, nevertheless, to the

same effective mass squared m_{eff}^2 , but only the solution $\mu'_g = \omega - \omega'$ is compatible with Eq. (6.29). By making the replacement $\omega' \rightarrow \bar{\omega}$ in Eq. (6.29), we can treat the massive gluonic excitation $\bar{\omega}$ in the same way as ω' . In the excitation $\alpha_s \cdot \bar{\omega}$ the effective mass \bar{m}_{eff}^2 appears not only through the corresponding gluon mean number, but in a more complicated way, see Eq. (5.2). Conventionally, we denote its "chemical potential" as $\alpha_s \cdot \bar{\mu}_g$. In principle, we can interpret our effective massive excitations as the gluon "flavors", but better to use the term "gluonic/bosonic species". As mentioned above, we can treat our massive excitations/species as some kind of quasi-particles, created by the self-interaction of massless gluon modes at non-zero temperature, i.e., consisting of the pure GM only.

Unfortunately, the interpretation of the effective gluonic excitations ω_{eff} , ω'_{eff} and $\bar{\omega}_{eff}$ is not so straightforward. In this connection, let us point out that the mass gap approach to QCD [5] requires the dominance of purely transversal virtual gluon fields with low-frequency components (or, equivalently, large-scale amplitudes) in its vacuum. This makes it possible to confine gluons to the vacuum even at non-zero temperature after the renormalization program for the mass gap is performed. Such strongly coupled gluon fields may form different types of stable field configurations with minimum energy (or even without it), such as flux-tube, which become string-type between heavy objects (like high-lying glueballs, for example), gluon chains, etc. Quite possible that the above mentioned effective gluonic excitations reflect the presence of such non-trivial gluon fields configurations as the relevant degrees of freedom in the YM vacuum at finite temperature as well [24–26]. Despite of these interpretations (which are present above as a subject for discussion only) or any other possible ones, however, an important thing has to be made perfectly clear. All our effective gluonic excitations are expressed in terms of the mass gap. It is dynamically generated by the strong self-interaction of massless gluon modes, and thus is mainly responsible for all the NP effects in the YM ground-state at any temperature [5]. It is the only one which determines the scale of the confining dynamics in the GM. All these effective gluonic excitations, which form a confining phase there, are of the NP origin. They vanish from GM spectrum in the PT $\Delta^2 = 0$ limit. So one can conclude that, in fact, (6.27) is the Hagedorn expansion in terms of the mass gap.

The maximum of temperature at which the Hagedorn expansion is valid is T_c , then it makes sense to identify T_c with the Hagedorn transition temperature T_h , i.e., to put $T_h = T_c$ within our approach (in agreement with recent lattice result [26] and see discussion in section VIII as well). It is worth underlying once more that the Hagedorn expansion (6.27) is nothing else but the gluon pressure (5.1) in the low-temperature region $T \leq T_h = T_c$. Its characteristic features are: a non-analytical dependence on the mass gap Δ^2 in some terms $\sim \Delta^2(\Delta^2)^{1/2}T \sim \Delta^3T$ and $\sim (\Delta^2)^{1/2}T^3 \sim \Delta T^3$. The PT correction of the α_s -order depends on the mass gap squared analytically. The presence of terms $\sim T^4$, the so-called SB-type terms, though overall coefficient in front of them vanishes in the PT $\Delta^2 = 0$ limit due to the initial normalization condition of the free PT vacuum to zero.

It is instructive to show this expansion as a function of the variable T_c/T . It suffices to do this by introducing the corresponding number of the exponents ν_i , $i = 1, 2, 3, 4, 5$. Their numerical values can be finally restored (if necessary) from the numerical values of the mass gap and T_c (see above). Using relations (3.5), (3.6), (3.8) and (6.28), for example $\omega_{eff} = 1.48\Delta = \nu_1 T_c$ which yields $\nu_1 = 3.75$, and so on. Such kind of the expansion looks like

$$\begin{aligned}
P_g(T) = & \frac{6}{\pi^2} \Delta^2 T^2 \sum_{n=1}^{\infty} \frac{1}{n^2} \left[\left(1 + \frac{n\omega_{eff}}{T} \right) e^{-n\nu_1(T_c/T)} - 4 \left(e^{-n\nu_4(T_c/T)} - e^{-n\nu_2(T_c/T)} \right) - e^{-n\nu_3(T_c/T)} \right] \\
& + \frac{16}{\pi^2} T^4 \sum_{n=1}^{\infty} \frac{1}{n^2} \left[\left(\frac{2}{n^2} + \frac{2\omega'_{eff}}{nT} + \frac{\omega'^2_{eff}}{T^2} \right) e^{-n\nu_4(T_c/T)} - \left(\frac{2}{n^2} + \frac{2\omega_{eff}}{nT} + \frac{\omega^2_{eff}}{T^2} \right) e^{-n\nu_1(T_c/T)} \right] \\
& - \frac{16}{\pi^2} T^4 \sum_{n=1}^{\infty} \frac{1}{n^2} \left[\left(\frac{2}{n^2} + \frac{2m'_{eff}}{nT} + \frac{m'^2_{eff}}{T^2} \right) e^{-n\nu_2(T_c/T)} + \left(\frac{2}{n^2} + \frac{2\bar{m}_{eff}}{nT} + \frac{\bar{m}^2_{eff}}{T^2} \right) e^{-n\nu_3(T_c/T)} \right] \\
& + \frac{9}{2\pi^2} \alpha_s \Delta^2 T^2 \sum_{n=1}^{\infty} \frac{1}{n^2} \left[\left(1 + \frac{n\bar{\omega}_{eff}}{T} \right) e^{-n\nu_5(T_c/T)} + \frac{n^2 \bar{m}^2_{eff}}{2T^2} \text{Ei}(-n\nu_5(T_c/T)) \right], \quad T \leq T_c, \quad (6.30)
\end{aligned}$$

Close to T_c the expansion for $P_g(T)$ can be obtained from the expression (6.30) by putting $T = T_c - \delta T$ and expanding in powers of a small $\delta = -1 + (T_c/T)$ in the $T \rightarrow T_c$ limit. So it shows an exponential rise in the number of dynamical degrees of freedom in this limit, explicitly seen in Fig. 1. In the opposite $T \rightarrow 0$ limit the gluon pressure is exponentially suppressed. Concluding this part, let us stress that the Hagedorn pressure (6.27), and hence (6.30), are closely related to the asymptotic of the gluon mean number (6.1) in the low-temperature region $T \leq T_c = T_h$. It is even possible to say that the Hagedorn structure of the expansion (6.27) is determined by the gluon mean number expansion in this temperature interval within the mass gap approach to QCD at finite temperature.

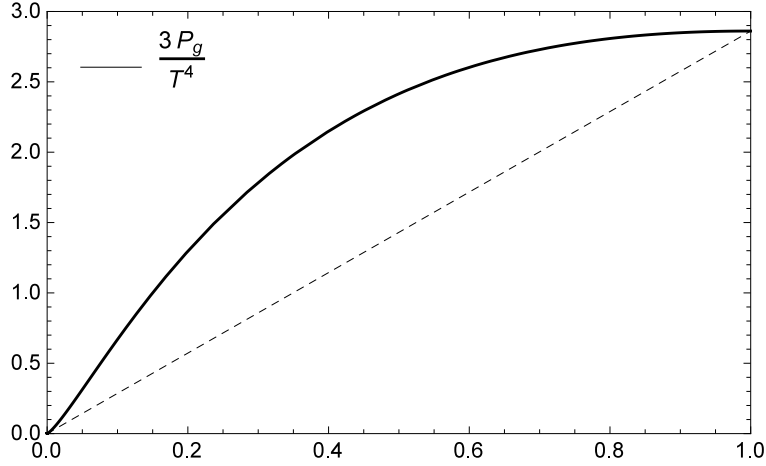


FIG. 2: The gluon pressure (7.1) scaled (i.e., divided) by $T^4/3$ is shown as a function of $(T_c/T)^2$ (solid curve). The exact T^2 behavior is shown as a straight dashed line. It is clearly seen that the gluon pressure substantially deviates from the exact T^2 one. The solid curve approaches a straight dashed line only in the limit of very high temperature, starting approximately from $(T_c/T)^2 \leq 0.04$, which corresponds to $T \geq 5T_c$ in agreement with Fig. 1 and Eq. (7.6).

VII. HIGH-TEMPERATURE EXPANSION. THE POLYNOMIAL STRUCTURE

In order to investigate the behavior of the gluon pressure (5.1) in the high-temperature region ($T \geq T_c$), it is convenient to re-write it as follows:

$$P_g(T) = P_{NP}(T) + P_{PT}^s(T) = \Delta^2 T^2 - \frac{6}{\pi^2} \Delta^2 P_1'(T) + \frac{16}{\pi^2} T M(T) + P_{PT}^s(T), \quad (7.1)$$

where

$$P_1'(T) = \int_0^{\omega_{eff}} d\omega \, \omega \, N_g(\beta, \omega) = \int_0^{\omega_{eff}} d\omega \frac{\omega}{e^{\beta\omega} - 1}. \quad (7.2)$$

It is easy to show that the expressions (5.1) and (7.1) are the same, because of the relations $P_1'(T) = (\pi^2/6)T^2 - P_1(T)$, $\int_0^\infty (d\omega \omega / e^{\beta\omega} - 1) = (\pi^2/6)T^2$, where the integral $P_1(T)$ is explicitly given in Eq. (3.2). At moderately high temperatures up to approximately a few T_c the exact functional dependence on the mass gap Δ^2 and temperature T of the gluon pressure (5.1) or, equivalently, (7.1) remains rather complicated. From Fig. 1 it follows that the NP effects due to the mass gap are still important up to rather high temperature, estimated as $(4-5)T_c$. The gluon pressure has a polynomial character in integer powers of T up to T^2 at very high temperatures (see below). As mentioned above, it is related to the corresponding asymptotic of the gluon mean number (3.7). In the high-temperature limit $T \rightarrow \infty$ ($\beta = T^{-1} \rightarrow 0$), the gluon mean number $N_g(\beta, \omega)$ can be reproduced by the corresponding series in powers of $(\beta\omega)$ if the variable ω is restricted, namely

$$N_g(\beta, \omega) = \frac{1}{e^{\beta\omega} - 1} = (\beta\omega)^{-1} \left[1 - \frac{1}{2}(\beta\omega) + O(\beta^2) \right], \quad \beta \rightarrow 0. \quad (7.3)$$

Let us note in advance that in what follows for our purpose it is sufficient to keep only the positive powers of T in the evaluation of the high-temperature expansion for the gluon pressure (7.1). Let us also remind that calculated finally terms not depending on temperature should be omitted, by definition [9]. Omitting all these tedious but rather simple derivations, which can be explicitly found in [5], the high-temperature expansion for the gluon pressure (7.1) up to the leading and next-to-leading orders, is as follows:

$$\begin{aligned}
P_g(T) \sim & \frac{12}{\pi^2} \Delta^2 \omega_{eff} T + \frac{8}{3\pi^2} \omega_{eff}^3 T \ln \left(\frac{\omega'_{eff}}{\bar{\omega}_{eff}} \right)^2 \\
& + \frac{2\sqrt{3}}{\pi^2} \Delta^3 T \arctan \left(\frac{2\omega_{eff}}{\sqrt{3}\Delta} \right) - \frac{16\sqrt{3}}{\pi^2} \Delta^3 T \arctan \left(\frac{\omega_{eff}}{\sqrt{3}\Delta} \right) \\
& + \frac{9}{2\pi^2} \alpha_s \Delta^2 \left[\frac{\pi^2}{6} T^2 - T \left(\Lambda_{YM} - \frac{\sqrt{3}}{2} \Delta \arctan \left(\frac{2\Lambda_{YM}}{\sqrt{3}\Delta} \right) \right) \right], \quad T \rightarrow \infty,
\end{aligned} \tag{7.4}$$

where $\bar{\omega}_{eff} = \sqrt{\omega_{eff}^2 + (3/4)\Delta^2}$ and here it is more convenient to express the gluon pressure in terms of effective ω 's (3.8), (6.28) and mass gap itself. A non-analytical dependence on the mass gap occurs in terms $\sim (\Delta^2)^{3/2} T \sim \Delta^3 T$, though Δ^2 is not an expansion parameter like α_s is in hot PT QCD, where a non-analytical dependence on α_s has been discovered (see, for example [27] and references therein). The term $\sim T^2$ has been first introduced in the phenomenological EoS [28] and widely discussed in [4, 5, 29–36]. On the contrary, in our approach both terms $\sim T^2$ and $\sim T$ have not been introduced by hand. They naturally appear on a general ground as a result of the explicit presence of the mass gap from the very beginning in the NP analytical EoS (7.1). It is interesting to note that the mass scale parameter in the leading NP term $\sim T^2$ in the expansion (7.4) is $(9/2\pi^2) \times (\pi^2/6)\Delta^2 = (3/4)\Delta^2 = \bar{m}_{eff}^2$, by definition in Eq. (3.6). Its numerical value is $\bar{m}_{eff} = 585$ MeV. The scale of the NP dynamics investigated in [28] is $M = 596$ MeV at almost the same T_c as ours, namely $T_c = 270$ MeV. It may or may not be a coincidence, but these numbers are very close to each other, though obtained by different approaches.

A few important issues concerning the high-temperature asymptotic of the gluon pressure (7.4) are to be discussed in more detail. The corresponding expansion for the composition $(16/\pi^2)TM_1(T) = (16/\pi^2)T[P_2(T) - P_4(T)]$, which enters the composition (3.3), is as follows:

$$\begin{aligned}
\frac{16}{\pi^2} TM_1(T) \sim & -2P_{SB}(T) + 2P_{SB}(T) - \Delta^2 T^2 + \frac{6}{\pi^2} \Delta^2 \omega_{eff} T - \frac{16}{\pi^2} TP_4^{(2)}(T) \\
\sim & -\Delta^2 T^2 + \frac{6}{\pi^2} \Delta^2 \omega_{eff} T - \frac{16}{\pi^2} TP_4^{(2)}(T), \quad T \rightarrow \infty,
\end{aligned} \tag{7.5}$$

where the expression for the integral $P_4^{(2)}(T)$ is not important for present discussion. So one can conclude that at high temperature the exact cancelation of the $P_{SB}(T)$ terms occurs within this composition. On the other hand, substituting it into Eq. (7.1) the cancelation of the $\Delta^2 T^2$ term occurs within the NP pressure $P_{NP}(T)$ itself. As it follows from Fig. 2 the both cancellations take place rather close to T_c , and the gluon pressure demonstrates highly non-trivial dependence on the mass gap and temperature up to rather high temperature. It approaches the exact T^2 limit at very high temperature only, starting approximately from $T \geq 5T_c$.

In a more compact form the previous expansion (7.4) looks like

$$P_g(T) \sim \alpha_s (3/4) \Delta^2 T^2 + [B_3 \Delta^3 + GeV^3] T, \quad T \rightarrow \infty, \tag{7.6}$$

where the first leading term, which analytically depends on the mass gap $\Delta^2 = (4/3)\bar{m}_{eff}^2$, comes from the PT part of the gluon pressure (more precisely from the NP part which is the α_s -order suppressed). The explicit expressions for both constants B_3 and GeV^3 (which becomes zero in the PT $\Delta^2 = 0$ limit) can be easily restored from the expansion (7.4), if necessary.

Concluding, a few remarks are in order to make. Let us emphasize once more that the SB term disappears from the NP gluon pressure (7.1) above T_c due to the normalization of the free PT vacuum to zero from the very beginning. The cancellation of the truly NP terms $\Delta^2 T^2$ simply shows that direct T^2 behavior cannot start just from T_c due to rather complicated dependence of the gluon pressure on the mass gap and temperature in the moderately high temperature interval (approximately up to $5T_c$, see Figs. 1 and 2). It would be very surprised if a pure NP contribution were survived in the limit of high temperature, while for its PT counterpart/correction it would be expected/possible. In other words, the $\Delta^2 T^2$ behavior of $P_g(T)$ in (7.1) is replaced by $\sim \alpha_s \Delta^2 T^2$ behavior in (7.4) or, equivalently, in (7.6) in this limit. At the same time, the second purely NP term $\sim T$ is suppressed in comparison with the first PT term in the high temperature limit in Eq. (7.6), indeed. Nevertheless, the approximate $\sim T^2$ behavior up to rather high temperature of such thermodynamic quantity as the trace anomaly or, equivalently, the interaction measure $I(T) = \epsilon(T) - 3P(T)$, which is very sensitive to the truly NP effects, is still possible. In other words, the deviation of $I(T)$ from the straight line can be not so big, unlike in Fig. 2. But it should be calculated in terms of the full pressure (see discussion below at the end of section VIII).

VIII. DISCUSSION AND CONCLUSIONS

The gluon pressure (5.1) has a few remarkable features. First of all, below T_c it is exponentially suppressed in the $T \rightarrow 0$ limit, see expansions (6.27) and (6.30) in this limit. Its the most important feature is that at low temperatures $T \leq T_c$ it is nothing else but the Hagedorn-type exponential series (6.27) for the effective gluonic excitations, which are expressed in terms of the mass gap, generated in its turn by the strong self-interaction of massless gluon modes. It is the only one which determines the confining dynamics in the GM [5]. The Hagedorn pressure of the glueball gas model is associated with a sum over a number of single noninteracting, relativistic particle species of the corresponding masses (low-lying glueballs) [8, 26, 37]. However, it alone was unable to correctly describe the corresponding thermodynamics lattice data below T_c , see for example [26, 34]. Only adding the closed bosonic string contribution [38], modelling the high-lying glueballs [20–23] exponential spectrum, success has been achieved. It is no a coincidence that $SU(3)$ lattice entropy density has been so nicely reproduced down to $0.7T_c$ by taking into account the string-type, confining configurations of gluon fields in this joint approach (glueball gas model plus bosonic string) [26]. Our Hagedorn exponential series (6.27) are very similar to those which have been previously found in the Polyakov-Nambu-Jona Lasinio (PNJL) approach by the extremization of the gluonic contribution to the thermodynamic effective potential with Polyakov loop [39–41] (and references therein). The Polyakov loop contribution has been implemented into the initial NJL model [42–44] just in order to take into account the confinement dynamics there. As mentioned above in section VII its NP scale [28, 40, 41] is almost the same as ours.

The gluon pressure (5.1) has a maximum at some characteristic temperature $T = T_c = 266.5$ MeV, see Fig. 1, at which its exponential rise at $T \leq T_c$ is changed to fall off at $T \geq T_c$. The relevant degrees of freedom below T_c are the effective gluonic excitations of different dynamical origin, expressed in terms of the mass gap (some of these NP excitations can be treated as massive quasi-particles/gluonic (bosonic) species). Above T_c they cease to exist, and the relevant degrees of freedom become weakly-interacting but still NP objects, which pressure is dominated by the $\sim \alpha_s T^2 \Delta^2$ -order term, see Eq. (7.6). Such drastic change in the character of the transition at T_c indicates that it might be of the first-order for the derivatives of the full pressure (mentioned above and discussed below). Thus, T_c can be understood as the temperature of the phase transition from the GM to the gluon plasma (GP). The scale of the exponential increase determines the value of the Hagedorn temperature [45]. Just this happens in the expansion (6.30) in the $T \rightarrow T_c$ limit. This means that maximum of temperature at which the Hagedorn structure is valid is T_c , so one has to identify T_c with the Hagedorn transition temperature T_h . Indeed, there is no other choice than to put $T_h = T_c$ in the mass gap approach to QCD at non-zero temperature (it can be also true for finite density, but this requires a separate investigation elsewhere). In other word, the mass gap approach makes the Hagedorn-type exponential series to necessarily arise in hot QCD, see discussion in [37] as well (and references therein). In lattice $SU(3)$ thermodynamics [34] the critical temperature is $T_c = 0.629\sqrt{\sigma} = 264.2$ MeV for the square root of the string tension $\sqrt{\sigma} = 420$ MeV. Such a good agreement between our characteristic and lattice critical temperatures, namely $266.5/264.2 = 1.0087$, though obtained by completely different analytical and lattice methods, is a serious argument to equal our and lattice T_c to the Hagedorn transition temperature T_h . For other justified arguments to identify T_c with T_h or to equal them (exactly or approximately) in different gauge groups for thermal YM fields see [26, 37, 45–48]. For example, in string theory the ratio between T_h and lattice T_c is $T_h/T_c = 1.069(5)$ [26, 46].

The gluon pressure's (7.1) fall off just after T_c is not a simple polynomial-type one, see Figs. 1 and 2. It is due to its rather complicated dependence on the temperature and mass gap in the region of high temperatures up to approximately $(4 - 5)T_c$. So NP effects are still important within our approach in this temperature interval, i.e., the GP can be considered as still remaining strongly interacting medium in this region. Only in the limit of very high temperature $T \rightarrow \infty$ it can be considered as weakly interacting medium and the gluon pressure has a corresponding polynomial-type character, Eq. (7.6). Possessing so nice features, the NP gluon pressure (7.1) at first sight seems to have one unpleasant "defect". From Fig. 1 it clearly follows that it will never reach the general SB constant/limit at very high temperatures after T_c . However, that is not a surprise, since the SB term has been canceled in the gluon pressure from the very beginning due to the normalization condition of the free PT vacuum to zero, as discussed in some details in section VII. The gluon pressure (5.1) may change its exponential regime below T_c only in the close neighborhood of T_c in order for its full counterpart to reach the requested SB limit at high temperatures. The SB term cannot be added to Eq. (5.1), even multiplied by the corresponding $\Theta((T/T_c) - 1)$ -function. In this case the full pressure will get a jump at $T = T_c$, which is not acceptable. So some other term(s), multiplied by the corresponding $\Theta((T_c/T) - 1)$ -function, should be added as well in order to ensure a smooth transition across T_c for the full gluon pressure. These problems make the inclusion of the SB term into the gluon EoS (5.1) highly non-trivial in order to transform it into the full gluon EoS. Only after its inclusion into Eq. (5.1) in a self-consistent way, such obtained full equation can be called the GP pressure or, equivalently, the GP EoS, and denoted as $P_{GP}(T)$.

In this connection, one thing has to be made perfectly clear. The truly NP gluon pressure (5.1) will remain an important part of the full GP pressure. It is this which will determine the low-temperature dynamical structure of the full pressure and even will play a significant role in it rather far away from T_c . Let us emphasize once more that

the gluon pressure $P_g(T)$ (though determined in the whole temperature range), but being the NP part of the full pressure, is not obliged and cannot reach SB limit at very high temperature. It is the full pressure $P_{GP}(T)$, properly scaled, which is obliged to approach this thermodynamical limit, and should be a continuously growing function of temperature at any point of its domain from zero to infinity. Thus the above-discussed unpleasant "defect" is not a real defect at all: on the contrary, the NP gluon pressure (5.1) has a correct thermodynamic limit at very high temperatures (7.6). The NP effects cannot indeed survive in the regime of very high temperatures, which is governed by the SB pressure $P_{SB} = (8/45)\pi^2 T^4$ of non-interacting massless particles (an ideal gas limit of gluons).

The gluon pressure (5.1) satisfies to all the required thermodynamical limits. It is exponentially suppressed in the $T \rightarrow 0$ limit and has an exponential rise in the $T \rightarrow T_c$ limit. This exponential increase in the number of dynamical degrees of freedom or, equivalently, an exponential grows in the density of states in this limit is clearly seen in Fig. 1 (the same picture occurs in the string theory at the Hagedorn transition [49]). Due to its Hagedorn's nature (6.27) at $T \leq T_c$ the NP gluon pressure (5.1) describes the GM as a dense states of the effective gluonic excitations of various dynamical nature, which are expressed in terms of the mass gap only. Thus, it has a correct Hagedorn structure at $T \leq T_c = T_h$. It demonstrates rather complicated dependence on the mass gap and temperature up to approximately $(4-5)T_c$. In the limit of very high temperature $T \rightarrow \infty$ it has a polynomial behavior consistent with the SB limit. That is why it can serve as a basic equation for its transformation into the full GP EoS, as discussed above.

Concluding, the main message we would like to convey is that the Hagedorn structure of the pressure is of crucial importance to correctly understand and describe the GM dynamical content at low temperatures within any approach or model. In the forthcoming paper we will present a general formalism how to transform the gluon pressure (5.1) into the full GP EoS in a self-consistent way (i.e., not destroying the Hagedorn structure below T_c , providing the smooth transition across T_c and approaching to the SB limit above T_c from below). Completing this program, we will be able to analytically describe YM $SU(3)$ lattice thermodynamics [26, 31, 34, 50–54] (and references therein).

Acknowledgments

We have been especially encouraged by the collection of papers in the book [8], edited by J. Rafelski. We thank R. Pisarski for bringing our attention to paper [28]. Our thanks also go to T. Biró, T. Csörgő, P. Ván, G. Barnaföldi, A. Lukacs and J. Nyiri for useful discussions, remarks and help. V.G. and A.S. are grateful to V. Kiguradze, N. Partsvania for constant support and interest. V.G. and A.S. acknowledge the support by the Hungarian National Fund (OTKA) 77816 and 31520 (P. Lévai). Partial support comes from "NewCompStar", COST Action MP1304. M.V. was also supported by the János Bolyai Research Scholarship of the Hungarian Academy of Sciences.

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- [1] *QUARK MATTER 2014*, Proc. of the XXIV Inter. Conf. on Ultra-Relativistic Nucleus-Nucleus Collisions, edited by: P. Braun-Munzinger, B. Friman, J. Stachel, 19-24, May, 2014, Darmstadt, Germany; Nucl. Phys. A, 931 (2014) 1.
 - [2] J.M. Cornwall, R. Jackiw, E. Tomboulis, Phys. Rev. D, 10 (1974) 2428.
 - [3] G.G. Barnafoldi, V. Gogokhia, J. Phys. G: Nucl. Part. Phys., 37 (2010) 025003; arXiv:0708.0163.
 - [4] V. Gogokhia, M. Vasúth, J. Phys. G: Nucl. Part. Phys., 37 (2010) 075015; arXiv:0902.3901.
 - [5] V. Gogokhia, G.G. Barnafoldi, The Mass Gap and its Applications (World Scientific, 2013).
 - [6] A. Jaffe, E. Witten, Yang-Mills Existence and Mass Gap, <http://www.claymath.org/prize-problems/>, <http://www.arthurjaffe.com>
 - [7] R. Hagedorn, Nuovo Cim. Suppl., 3 (1965) 147.
 - [8] MELTING HADRONS, BOILING QUARKS, From Hagedorn Temperature to Ultra-Relativistic heavy-Ion Collisions at CERN, edited by J. Rafelski (Springer Open, 2015).
 - [9] J.I. Kapusta, C. Gale, Finite-Temperature Field Theory (Cambridge University Press, 2006).
 - [10] J. Letessier, J. Rafelski, Hadrons and Quark-Gluon Plasma (Cambridge University Press, 2004).
 - [11] L. Dolan, R. Jakiw, Phys. Rev. D, 9 (1974) 3320.
 - [12] M.E. Peskin, D.V. Schroeder, An Introduction to Quantum Field Theory (ABP, Westview Press, 1995).
 - [13] W. Marciano, H. Pagels, Phys. Rep. C, 36 (1978) 137.
 - [14] I.V. Andreev, Chromodynamics and Hard Processes at High Energie ("NAUKA", Moscow, 1981).
 - [15] P. Lévai, U. Heinz, Phys. Rev. C, 57 (1998) 1879; hep-ph/9710463.
 - [16] Ph. Boucaud, J.P. Leroy, J. Micheli, O. Pène, C. Roiesnel, hep-ph/9810437.
 - [17] PARTICLE DATA GROUP, J. Phys. G: Nucl. Part. Phys., 37 (2010) 101.
 - [18] I.S. Gradshteyn, I.M. Ryzhik, Tables of Integrals, Series, and Products (Academic Press, 2007).
 - [19] A.P. Prudnikov, Y. A. Brichkov, O.I. Marichev, Integrals and Series ("NAUKA", Moscow, 1981).
 - [20] V. Mathieu, A.K. Kochelev and V. Vento, Int. J. Mod. Phys. E, 18 (2009) 1; arXiv:0810.4453.
 - [21] D.H. Rischke, M.I. Gorenstain, A. Schäfer, H. Stöcker and W. Greiner, Phys. Lett. B, 278 (1992) 19.

- [22] H.B. Meyer, arXiv:hep-lat/0508002.
- [23] Y. Chen et al., Phys. Rev. D, 73 (2006) 014516.
- [24] N. Isgur, J.E. Paton, Phys. Rev. D, 31 (1985) 2910.
- [25] F. Buisseret, V. Mathieu, C. Semay, Phys. Rev. D, 80 (2009) 074021; arXiv:0906.3098.
- [26] H.B. Meyer, Phys. Rev. D, 80 (2009) 051502(R); arXiv:0905.4229.
- [27] K. Kajantie, M. Laine, K. Rummukainen and Y. Schroder, Phys. Rev. D, 67 (2003) 105008.
- [28] P.N. Meisinger, T.R. Miller, M.C. Ogilvie, Phys. Rev. D, 65 (2002) 034009; hep-ph/0108009.
- [29] R.D. Pisarski, Prog. Theor. Phys. Suppl., 168 (2007) 276; hep-ph/0612191.
- [30] P. Castorina, D.E. Miller, H. Satz, Eur. Phys. J. C, 71 (2011) 1673; arXiv:1101.1255.
- [31] M. Panero, Phys. Rev. Lett., 103 (2009) 232001; arXiv:0907.3719.
- [32] O. Andreev, Phys. Rev. D, 76 (2007) 087702; arXiv:0706.3120.
- [33] T.S. Biró, J. Cleymans, Phys. Rev. C, 78 (2008) 034902; hep-ph/9710463.
- [34] Sz. Borsanyi, G. Endrodi, Z. Fodor, S.D. Katz, K.K. Szabo, JHEP, 07 (2012) 056; arXiv:1204.6184.
- [35] E. Shuryak, Prog. Part. Nucl. Phys., 62 (2009) 48; arXiv:0807.3033.
- [36] E. Megías, E.R. Arriola and L.L. Salcedo, Phys. Rev. D, 80 (2009) 056005; arXiv:0903.1060.
- [37] F. Buisseret, G. Lacroix, Phys. Lett. B, 705 (2011) 405; arXiv:1105.1092.
- [38] B. Zweibach, A First Course in String Theory (Cambridge University Press, 2009).
- [39] K. Fukushima, Phys. Lett. B, 591 (2004) 277; arXiv:hep-ph/0310121.
- [40] H. Nishimura, M. C. Ogilvie, Phys. Rev. D, 81, (2010) 014018; arXiv:0911.2696.
- [41] H. Nishimura, M. C. Ogilvie, K. Pangeni, arXiv:1503.00060.
- [42] Y. Nambu, G. Iona-Lasinio, Phys. Rev., 122 (1961) 345.
- [43] S.P. Klevansky, Rev. Mod. Phys., 64 (1992) 649.
- [44] T. Hatsuda, T. Kunihiro, Phys. Rept., 247 (1994) 221; hep-ph/9401310.
- [45] J. Cleymans, D. Worku, Mod. Phys. Lett. A, 26 (2011) 1197; arXiv:1103.1463.
- [46] B. Lucini, M. Teper, U. Wenger, J. High Energy Phys., 01 (2004) 061.
- [47] B. Bringoltz, M. Teper, Phys. Rev. D, 73 (2006) 014517; hep-lat/0508021.
- [48] J. Braun, A. Eichhorn, H. Gees, J.M. Pawłowski, Eur. Phys. J. C, 70 (2010) 689; arXiv:1007.2619.
- [49] J.J. Atick, E. Witten, Nucl. Phys. B, 310 (1988) 291.
- [50] G. Boyd, J. Engels, F. Karsch, E. Laermann, C. Legeland, M. Lutgemeier, B. Petersson, Nucl. Phys. B, 469 (1996) 419; hep-lat/9602007.
- [51] M. Fukugita, M. Okawa, A. Ukawa, Phys. Rev. Lett., 63 (1989) 1768.
- [52] B. Lucini, M. Teper, U. Wenger, J. High Energy Phys., 02 (2005) 033; hep-lat/0502003.
- [53] S. Datta, S. Gupta, Phys. Rev. D, 82 (2010) 114505; arXiv:1006.0938.
- [54] B. Beinlich, F. Karsch, A. Peikert, Phys. Lett. B, 390 (1997) 268.